

# HODGE STRUCTURES ORBIFOLD HODGE NUMBERS AND A CORRESPONDENCE IN QUASITORIC ORBIFOLDS

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**ABSTRACT.** We give Hodge structures on quasitoric orbifolds. We define orbifold Hodge numbers and show a correspondence of orbifold Hodge numbers for crepant resolutions of quasitoric orbifolds. In short we extend Hodge structures to a non complex non almost complex setting .

## 1. Introduction

The purpose of this paper is to extend Hodge structures in a non complex non almost complex setting. First we give a canonical Hodge structure (see 4.7) to quasitoric orbifolds. We compute Hodge numbers. Then as an application we define orbifold Hodge numbers and show a correspondence of these numbers for crepant resolutions. Since toric manifolds and possibly orbifolds are used widely in physics as target spaces of non linear sigma models, with canonical Hodge structures possible in quasitorics we feel these spaces can also be put into some consideration and inspection by experts of the above areas. Also this is an extension of Deligne's mixed Hodge structures in a non-complex algebraic setting thus will draw interest of Mathematicians. During the process we also stumble on a short proof of Hodge numbers of Projective toric orbifolds where its Hodge structure is Deligne's Hodge structures.

Physicists believe that orbifold string theory is equivalent to ordinary string theory of certain desingularizations. This belief motivated a body of conjectures, collectively referred to as the orbifold string theory conjecture. The conjecture we are interested in is the K-orbifold string theory conjecture. It states that there is a natural isomorphism between the orbifold K-theory of a Gorenstein orbifold and the ordinary K-theory of its crepant resolution. To construct a natural isomorphism as the conjecture demands, is a very hard problem. But many weaker versions of the conjecture that compare Euler numbers, Hodge numbers, etc. has been studied extensively in the literature in the case of algebraic orbifolds and orbifolds with global quotients. Batyrev and Dais in [2] Poddar and Lupercio in [9] and Yasuda in [16], proved in particular the equality of orbifold Hodge numbers of smooth crepant resolutions for Gorenstein algebraic orbifolds and for the non-Gorenstein algebraic case was proved by Yasuda [18]. We generalize this correspondence to a non-algebraic non- analytic and non-global setting. The correspondence also generalizes the string

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theoretic Hodge number Correspondence of Batyrev and Dias to a non- algebraic non-analytic setting.

A quasitoric orbifold (defined in [6], [12]) is a  $2n$ -dimensional differentiable orbifold equipped with a smooth action of the  $n$ -dimensional compact torus such that the orbit space is diffeomorphic as manifold with corners to a  $n$ -dimensional simple polytope (A  $n$ - dimensional polytope is called simple if every vertex is the intersection of exactly  $n$  codimension one faces). The preimage of every codimension one face is a torus invariant  $(2n - 2)$ -dimensional quasitoric orbifold which is stabilized by a circle subgroup of the form  $\{(e^{2\pi a_1 t}, \dots e^{2\pi a_n t}) : t \in \mathbb{R}\}$ . The vector  $(a_1, \dots a_n)$  is a primitive integral vector called the characteristic vector associated with this codimension one face. In general a codimension  $k$  face is the intersection of  $k$  codimension one faces and its characteristic set consists of the characteristic vectors of these codimension one faces. The characteristic set of every face is linearly independent over  $\mathbb{R}$ .

Quasitoric manifolds (and orbifolds, although not in full generality,) were introduced in [6] . They got their present name in [3]. They are generalizations of smooth projective toric varieties. They include manifolds which do not admit an almost complex structure such as  $CP^2 \sharp CP^2$ . A broader class of quasitoric orbifolds were defined in [12]. In that paper and the subsequent papers [7] and [8], questions relating to homology, cohomology, almost complex structures and equivariant blow-down maps were addressed. McKay correspondence of Chen-Ruan cohomology was also established for four and six dimensional quasitoric orbifolds in [7] and [8].

Thus quasitoric orbifolds give a vast family of orbifolds which are neither complex or almost complex. In this paper we address the question of Hodge structures of these orbifolds. Since  $CP^2 \sharp CP^2$  is a non- almost complex quasitoric orbifold it gets a Hodge structure in spite of being non-almost complex. This opens possibility of Hodge theory in a non almost complex setting. The idea of the proof is to relate the cohomology of a quasitoric orbifold with the cohomology of a projective toric orbifold which is a Kahler orbifold . The complex De Rham cohomology of the two spaces are shown to be isomorphic as a graded vector space and so the Hodge structure of one can be pulled back to the other.

With Hodge structures possible we can define orbifold Hodge numbers and get a correspondence for these numbers under crepant blowdown by imitating Batyrev-Dias correspondence.

A general quasitoric orbifold differs combinatorially from a projective toric orbifold in the following manner. In the case of a projective toric orbifold, the characteristic vector of a codimension one face of the orbit polytope is normal to that face. This enables the characteristic vectors to generate cones that fit together to form a toric fan. In a general quasitoric orbifold the normality condition is relaxed to linear independence of characteristic sets of faces.

The paper is organized in the following manner. In section 2 we give a combinatorial construction of quasitoric orbifolds. In section 3 we discuss Betti numbers of quasitoric orbifolds. In section 4 we define Hodge structures and provide a canonical Hodge structure by showing cohomological vector space isomorphism between a quasitoric and a projective toric. In section 5 as an application we define and show

orbifold Hodge number correspondence. In section 6 we give an argument to show every simple polytope is combinatorially equivalent to a rational polytope which is used to show cohomological isomorphism of a quasitoric with a projective toric.

## 2. Quasitoric orbifolds

In this section we describe the combinatorial construction of quasitoric orbifolds. Notations established in this section will be used later.

Take a copy  $N$  of  $\mathbb{Z}^n$  and form a torus  $T_N := (N \otimes_{\mathbb{Z}} \mathbb{R})/N \cong \mathbb{R}^n/N$ .

Take a submodule  $M$  of  $N$  of rank  $m$  and construct the torus  $T_M := (M \otimes_{\mathbb{Z}} \mathbb{R})/M$  of dimension  $m$ . Define the map  $\zeta_M : T_M \rightarrow T_N$  the obvious map generated by the inclusion map  $M \rightarrow N$ .

**Definition 2.1.** We define the image of  $T_M$  under the map  $\zeta_M$  as  $T(M)$ . If  $M$  is a sub-module of rank 1 and  $\lambda$  is the generator then we call the image  $T(\lambda)$ .

**Definition 2.2.** A polytope is  $P$  is a subset of  $\mathbb{R}^n$  which is diffeomorphic as manifolds with corners to a convex hull  $C$  of a finite number of points in  $\mathbb{R}^n$ . The faces of  $P$  are images of faces of  $C$ .

**Definition 2.3.** A simple polytope is a polytope where each vertex is an intersection of  $n$  co-dimension one faces which are in general position.

**Definition 2.4.** Codimension one faces of a polytope  $P$  are called facets. In a simple polytope every  $k$  dimensional face is an intersection of  $n-k$  facets. We call  $\mathcal{F} = \{F_1, F_2 \dots F_M\}$  the set of facets of the simple polytope  $P$ .

**Definition 2.5.** We define a map  $\Lambda : \mathcal{F} \rightarrow \mathbb{Z}^n$  where  $F_i$  is mapped to  $\Lambda(F_i)$  and if  $F_{i_1} \dots F_{i_k}$  intersect to form a face of the polytope  $P$  then the corresponding  $\Lambda(F_{i_1}) \dots \Lambda(F_{i_k})$  are linearly independent. From now onwards we call  $\Lambda(F_i)$  as  $\lambda_i$  and call it a characteristic vector and  $\Lambda$  the characteristic function.

**Remark 2.1.** In this article we consider only primitive characteristic vectors and call the corresponding quasitoric orbifolds as primitive quasitoric orbifolds. The codimension of the singular locus of these orbifolds is at least 4.

**Definition 2.6.** For a face  $F$  define  $\mathcal{I}(F) = \{i : F \subset F_i, F_i \in \mathcal{F}\}$ . The set  $\Lambda_F = \{\lambda_i : i \in \mathcal{I}(F)\}$  is called the characteristic set of  $F$ . We call  $N(F)$  be the submodule generated  $\Lambda_F$ . If  $\mathcal{I}(F)$  is empty  $N(F) = 0$ .

For any point  $p$  in the polytope we denote  $F(p)$  the face whose relative interior contains  $p$ . We define an equivalence relation in  $P \times T_N$  where  $(p, t_1) \sim (q, t_2)$  if  $p = q$  and  $t_2^{-1}t_1 \in T(N(F(p)))$  where  $N(F(p))$  is the sub module of  $N$  generated by integral linear combinations of vectors of  $\Lambda_{F(p)}$ . The quotient space  $X = P \times T_N / \sim$  has a structure of an  $2n$  dimensional orbifold and are called quasitoric orbifolds.

The pair  $(P, \Lambda)$  is a model for the above space. If vectors comprising  $\Lambda_F$  are unimodular for all faces  $F$  we get a quasitoric manifold. The  $T_N$  action on  $P \times T_N$  induces a torus action on the quotient space  $X$ , of the equivalence relation, and quotient of this action is the polytope  $P$ . Let us denote the quotient map by  $\pi :$

$X \rightarrow P$ .  $\pi^{-1}(w)$  for a vertex  $w$  of  $P$  is a fixed point of the above action and we will denote it by  $w$  without confusion.

**2.1. Orbifold structure.** For every vertex  $w$  in  $P$  consider open set  $U_w$  of  $P$  the complement of all faces not containing the vertex  $w$ . We define

$$(2.1) \quad X_w = \pi^{-1}(U_w) = U_w \times T_N / \sim .$$

For any face  $F$  containing the vertex  $w$  there is a natural inclusion of  $N(F)$  in  $N(w)$  and  $T_{N(F)}$  in  $T_{N(w)}$ . We define another equivalence relation  $\sim_w$  on  $U_w \times T_{N(w)}$  as follows.

For  $p \in U_w$ , let  $F$  be the the face which contains  $p$  in its relative interior, by definition  $F$  contains  $w$ . We define the relation as  $(p, t_1) \sim_w (q, t_2)$ , if  $p = q$  and  $t_2^{-1}t_1 \in T_{N(F)}$ . We define

$$(2.2) \quad \tilde{X}_w = U_w \times T_{N(w)} / \sim_w .$$

The above space is equivariantly diffeomorphic to an open set in  $C^n$  with the standard torus action on  $C^n$  and  $T_{N(w)}$  action on  $\tilde{X}_w$ . The diffeomorphism will be clear from the subsequent discussion. The map  $\zeta_{N(w)} : T_{N(w)} \rightarrow T_N$  induces a map from  $\zeta_w : \tilde{X}_w \rightarrow X_w$  in the following way

$$(2.3) \quad \zeta_w((p, t) \sim_w) = (p, \zeta_{N(w)}(t)) \sim .$$

The kernel of the map  $\zeta_{N(w)}$  is  $G_w = N/N(w)$  is a subgroup  $T_{N(w)}$  and has a smooth action on  $\tilde{X}_w$  and the quotient of this action is  $X_w$ . This action is not free and so  $X_w$  is an orbifold and the uniformizing chart of  $X_w$  is  $(\tilde{X}_w, G_w, \zeta_w)$ .

We define a homeomorphism  $\phi(w) : \tilde{X}_w \rightarrow \mathbb{R}^{2n}$  as follows. Assume without loss of generality  $F_1, F_2 \dots F_n$  are the facets containing  $w$  and  $p_i(w) = 0$  is the the facet  $F_i$  and in  $U_w$   $p_i$ 's have non-negative values with positive in interiors of  $U_w$ . Let  $\Lambda_w$  be the corresponding set of characteristic vectors represented as follows

$$(2.4) \quad \Lambda_w = [\lambda_1 \dots \lambda_n].$$

If  $q(w)$  be the representation of the angular coordinates of  $T_N$  in the basis with respect to  $\lambda_1 \dots \lambda_n$  of  $N \otimes_{\mathbb{Z}} \mathbb{R}$ . Then the standard coordinates  $q$  are related in the following manner to  $q(w)$

$$(2.5) \quad q = \Lambda_w q(w).$$

The homeomorphism  $\phi(w) : \tilde{X}_w \rightarrow \mathbb{R}^{2n}$  is

$$(2.6) \quad x_i = x_i(w) := \sqrt{p_i(w)} \cos(2\pi q_i(w)), \quad y_i = y_i(w) := \sqrt{p_i(w)} \sin(2\pi q_i(w)) \quad \text{for } i = 1, \dots, n.$$

We write

$$(2.7) \quad z_i = x_i + \sqrt{-1}y_i, \quad \text{and} \quad z_i(w) = x_i(w) + \sqrt{-1}y_i(w).$$

Now consider the action of  $G_w = N/N(w)$  on  $\tilde{X}_w$ . An element  $g$  of  $G_w$  is represented by a vector  $\sum_{i=1}^n a_i \lambda_i$  in  $N$  where each  $a_i \in \mathbb{Q}$ . The action of  $g$  transforms the coordinates  $q_i(w)$  to  $q_i(w) + a_i$ . Therefore

$$(2.8) \quad g \cdot (z_1, \dots, z_n) = (e^{2\pi\sqrt{-1}a_1} z_1, \dots, e^{2\pi\sqrt{-1}a_n} z_n).$$

We define

$$(2.9) \quad G_F := ((N(F) \otimes_{\mathbb{Z}} \mathbb{Q}) \cap N)/N(F).$$

We denote the space  $X$  with the above orbifold structure by  $\mathbf{X}$ .

**2.2. Invariant Suborbifolds.** The  $T_N$  invariant subset  $\pi^{-1}(F)$  where  $F$  is a face of  $P$  is a quasitoric orbifold. The face  $F$  acts as the polytope of  $\mathbf{X}(F)$  and the characteristic vectors are obtained by projecting characteristic vectors of  $\mathbf{X}$  to  $N/N(F)$  where  $N(F) = N(F) \otimes_{\mathbb{Z}} \mathbb{Q} \cap N$ . With this structure  $\mathbf{X}(F)$  is a suborbifold of  $\mathbf{X}$ . The suborbifolds corresponding to the facets are called characteristic suborbifolds. We denote the interior of a face by  $F^\circ$ . The interior of a vertex  $w^\circ$  is  $w$ .

**2.3. Orientation.** Quasitoric orbifolds are oriented. For more detailed discussion see section 2.8 of [8]. A choice of orientation of  $T_N$  and a choice of orientation of the polytope  $P$  induces an orientation of the quasitoric orbifold  $\mathbf{X}$ .

**2.4. Omniorientation.** A choice of orientations of the normal bundles of the orbifolds corresponding to the facets (which we named as characteristic suborbifolds) is termed as fixing an omniorientation. This is equivalent to fixing the sign of the characteristic vector associated to the facet (note: we call co-dimension one faces as facets). A quasitoric orbifold with a fixed omniorientation is called an **omnioriented quasitoric orbifold**. A quasitoric orbifold is **positively omnioriented** if it has an omniorientation such that for every vertex  $w$ ,  $\Lambda_w$  has a positive determinant. For more detailed discussion see section 2.9 of [8].

### 3. Betti numbers of a Quasitoric Orbifold

Poddar and Sarkar computed the  $\mathbb{Q}$  homology and cohomology of quasitoric orbifolds in [12]. In particular the computation of homology in Section 4 of [12] gives a strong connection between the combinatorics of the polytope  $P$  and the Betti numbers. We discuss the connection in following proposition.

**Proposition 3.1.** *Quasitoric orbifolds with combinatorially equivalent polytopes have same Betti numbers.*

*Proof.* A brief discussion of the homology computation in [12] is required to establish the above proposition. The computation depends on defining a continuous height function on the polytope  $P$  with following properties.

- (1) Distinguishes vertices.
- (2) Strictly increases or decreases on edges.
- (3) Each face has a unique maximum and minimum vertex.
- (4) The maximum vertex is the unique vertex of the face where all the edges of the face meeting the vertex has a maximum on the vertex.
- (5) The minimum vertex is the unique vertex of the face where all the edges of the face meeting the vertex has a minimum on the vertex.

A vertex distinguishing linear functional of  $\mathbb{R}^n$  does the job. Here we assume  $P$  is embedded in  $\mathbb{R}^n$ . Once we have such a function we orient the edges of the polytope in increasing direction of the height function and arrange the **vertices in increasing order of height**. We define index  $i_w$  of a vertex  $w$  as the number of incoming edges. The smallest face containing these incoming edges is the largest face  $F_w$  which has  $w$  as the maximum vertex. Now start attaching  $2i_w$   $q$ -cells following the increasing order of vertices. The  $q$ -cell covers the entire inverse image of  $F_w$  in the orbifold. For definition and description of  $q$ -cells and the attaching maps we ask the **reader to consult** [12].

Now each face has a unique maximum vertex  $w$  and interior of the face will be contained in  $F_w$  by points 3 and 4 above. So each face gets covered and each point in the orbifold is in the interior of exactly one  $q$ -cell. Considering the polytope as a face there will be exactly one 0  $q$ -cell and one  $2n$   $q$ -cell. Thus we get a  $q$  cellular decomposition of the quasitoric orbifold.

Now it is shown in [12] that the  $2k$  Betti numbers depends on the number of vertices with index  $k$  while the odd Betti numbers are zero. Now if we have two quasitoric orbifolds with two combinatorially equivalent polytopes (which means they are diffeomorphic as manifold with corners) the height function of one composed with the diffeomorphism gives a height function of the other with identical vertex indices. Thus their Betti numbers will be same by what is done in [12].  $\square$

**Corollary 3.2.** *The dimension of each degree of  $\mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  singular cohomology of a quasitoric orbifolds  $\mathbf{X}$  and  $\mathbf{X}'$  with combinatorially equivalent polytopes are same .*

*Proof.* By Universal coefficient theorem.  $\square$

**Corollary 3.3.** *The dimension of each degree of  $\mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  singular cohomology of a quasitoric orbifold  $\mathbf{X}$  is identical with a projective toric orbifold  $X'$  .*

*Proof.* Take a quasitoric orbifold  $\mathbf{X}$ . A slight perturbation makes the polytope  $P$  associated with the orbifold into a rational polytope (see section 5.1.3.in [3] or see appendix) without changing its combinatorial class, and with suitable dilations makes it into an integral polytope  $P'$  which is combinatorially equivalent to  $P$ . Now from  $P'$  taking normal fan we get a projective toric orbifold  $X'$  (the analytic structure determines the orbifold structure so we do not use the bold notation) with polytope  $P'$ . Since polytope  $P$  and polytope  $P'$  are combinatorially equivalent by (3.2) the above holds.  $\square$

**Corollary 3.4.** *Each degree of the cohomology of the two spaces are isomorphic.*

*Proof.* Since they have the same dimension and the dimensions are finite so the vector spaces are isomorphic. We define the isomorphisms as  $J_k$  where  $k$  is the degree of the cohomology.  $\square$

#### 4. Hodge Structure

**Definition 4.1.** *A pure Hodge structure of weight  $n$  consists of an Abelian group  $H_K$  and a decomposition of its complexification into complex subspaces  $H^{p,q}$  where*



$p + q = n$  with the property conjugate of  $H^{p,q}$  is  $H^{q,p}$ .

$$(4.1) \quad H_C = H_K \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}.$$

and

$$(4.2) \quad \overline{H^{p,q}} = H^{q,p}.$$

**Definition 4.2.** By a Hodge structure on a compact space we mean the singular cohomology group of degree  $k$  has a pure Hodge structure of weight  $k$  for all  $k$ .

**Proposition 4.1.** Kahler compact orbifolds have a canonical Hodge structure.

*Proof.* By Baily's Hodge decomposition see [1]. □

**Proposition 4.2.** Projective toric orbifolds coming from integral simple polytopes are Kahler.

*Proof.* By theorem 8.1 9.1 and 9.2 in [14]. □

**Corollary 4.3.** Projective toric orbifolds coming from integral simple polytopes have a canonical Hodge structure.

**Definition 4.3.** Let  $H^{p,q}$  be the  $(p, q)$  Hodge component of the canonical Hodge structure on Projective toric orbifolds  $X'$  coming from integral simple polytopes. We define

$$(4.3) \quad H^{p,q}(X') = H^{p,q}.$$

and

$$(4.4) \quad h^{p,q}(X') = \dim(H^{p,q}(X')).$$

Let  $\mathbf{X}$  be a quasitoric orbifold and  $X'$  be the projective toric orbifold whose integral simple polytope  $P'$  is combinatorially equivalent to the polytope  $P$  of  $\mathbf{X}$ . We assign

$$(4.5) \quad H^{p,q}(\mathbf{X}) = J_k(H^{p,q}(X')).$$

where  $p + q = k$  and  $J_k$  is the isomorphisms of the degree  $k$  cohomologies defined in (3.4).

**Theorem 4.4.** The above assignment defines a Hodge structure on  $\mathbf{X}$  depending on  $J_k$ . For independence of  $J_k$  see 4.7.

*Proof.* By above and corollary (3.3). □

**Theorem 4.5.** The assignment does not depend on  $X'$ .

*Proof.* To show the above we must understand the  $E$ -polynomial. Let  $Y$  be an algebraic variety over  $\mathbb{C}$  which is not necessarily compact or smooth. Denote by  $h^{p,q}(H_c^k(Y))$  the dimension of the  $(p, q)$  Hodge component of the  $k$ -th cohomology with compact supports. This is a generalization of the Hodge structures discussed

on the above class of compact projective toric orbifolds and are called mixed Hodge structures. For more detailed discussion we ask the reader to consult [15].

We define

$$(4.6) \quad e^{p,q}(Y) = \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(Y)).$$

The polynomial

$$(4.7) \quad E(Y; u, v) := \sum_{p,q} e^{p,q}(Y) u^p v^q$$

is called  $E$ -polynomial of  $Y$ . When we have a proper Hodge structure like the above class of compact projective toric orbifold  $X'$ ,

$$(4.8) \quad e^{p,q}(X') = (-1)^{p+q} h^{p,q}(X').$$

Now if we have a stratification of an algebraic variety  $Y$  by disjoint locally closed sub-varieties  $Y_i$  (i.e  $Y_i \subset Y$  and  $Y = \cup_i Y_i$ ) by proposition 3.4 of [2]

$$(4.9) \quad E(Y; u, v) = \sum_i E(Y_i; u, v)$$

Now in a projective toric orbifold coming from a integral simple polytope as in our case we have a stratification by algebraic tori corresponding to the interior of each face. Let  $X'$  be the concerned orbifold and  $F_i$  be a  $k$  dimensional face of the corresponding polytope then  $\pi^{-1}(F_i^\circ)$  is a  $k$  dimensional algebraic tori which we denote  $X'_i$ . So by (4.9) we have

$$(4.10) \quad E(X'; u, v) = \sum_i E(X'_i; u, v).$$

**Here  $i$  runs over all the faces.** Now if we have two projective toric orbifolds  $X'$  and  $X''$  both having combinatorially equivalent polytopes with that of  $\mathbf{X}$  by (4.10) we claim they have the same  $E$ -polynomial. This is because since they have combinatorially equivalent polytopes, number of faces of a given dimension will be same for each polytope. So the sum on the right hand side of (4.10) can be partitioned into  $E$ -polynomial of  $k$  dimensional algebraic tori with a multiplicity of number of faces of dimension  $k$ , where  $k$  runs from 0 to dimension of the polytopes. Since same dimensional algebraic tori have same  $E$ -polynomial the above claim holds.

Thus the Hodge numbers of the two projective toric orbifolds will be same by (4.8). So the theorem holds.  $\square$

**Theorem 4.6.** *The Hodge numbers of a quasitoric orbifold are as follows*

$$h^{p,q}(\mathbf{X}) = 0 \text{ if } p \neq q \text{ and } h^{p,p}(\mathbf{X}) = \dim(H^{2p}(\mathbf{X}, \mathbb{C})).$$

*Proof.* We show this for projective toric orbifolds coming from integral simple polytope. We know that the  $E$ -polynomial of a  $k$ -dimensional algebraic torus is  $(uv - 1)^k$ . Since by (4.10) the  $E$ -polynomial of the projective toric orbifold decomposes into sum of  $E$ -polynomial of algebraic tori and since  $E$ -polynomial of the algebraic tori have only terms of the form  $(uv)^l$ , implies that coefficient of  $u^p v^q$  is zero if  $p \neq q$  in the  $E$ -polynomial of the projective toric orbifold. Since these projective toric orbifolds have a proper Hodge structure the claim of the theorem is true.  $\square$

**Theorem 4.7.** *The above Hodge structure does not depend on  $J_k$  and is canonical.*



*Proof.* Since there is only one non-zero Hodge number for a given degree of cohomology, different  $J_k$  will define the same Hodge decomposition.  $\square$

**Remark 4.8.** *The above proof of Hodge numbers of quasitoric orbifolds is also a proof for Hodge numbers of projective toric orbifolds coming from Deligne's mixed hodge structures. We have not seen this proof in literature before.*

**4.1. Example.** We compute the Hodge structure for  $CP^2 \sharp CP^2$  which does not have an almost complex structure. We take a projective toric orbifold  $X'$  with a combinatorially equivalent polytope  $P'$ . Since the polytope of  $P$  is a four sided polygon (see example 1.19 [6]) it will have four vertices, four edges and one 2-face.

$$(4.11) \quad E(X'; u, v) = (uv - 1)^2 + 4(uv - 1) + 4.$$

$$(4.12) \quad E(X'; u, v) = u^2v^2 + 2uv + 1.$$

This tallies with the cohomology of  $CP^2 \sharp CP^2$  and so we have the decomposition  $h^{2,2} = 1$ ,  $h^{1,1} = 2$  and  $h^{0,0} = 1$ ,

## 5. An application -Orbifold Hodge numbers and a correspondence

**5.1. Orbifold Hodge numbers.** Orbifold Hodge numbers for closed global quotient orbifold was defined in [13] and [2] and for Kahler orbifolds in [11]. They are the dimensions of the Dolbeaut orbifold cohomology (see [11] section 2.2). They depend on the twisted sectors of the orbifold. The twisted sector for toric variety was computed in [10]. The determination of twisted sectors of quasitoric orbifolds are similar in essence. Let  $\mathbf{X}$  be an omnioriented quasitoric orbifold (i.e the signs of characteristic vectors are fixed). Consider an element  $g$  belonging to the group  $G_F$  defined in equation (2.9). Then  $g$  may be represented by the vector  $\sum_{j \in \mathcal{I}(F)} a_j \lambda_j$  where  $a_j$  is restricted to  $[0, 1) \cap \mathbb{Q}$  and  $\lambda_j$  belongs to the characteristic set of  $F$ . We define the degree shifting number or **age** as

$$(5.1) \quad \iota(g) = \sum a_j.$$

For faces  $F$  and  $H$  of  $P$  we write  $F \leq H$  if  $F$  is a sub-face of  $H$ , and  $F < H$  if it is a proper sub-face. If  $F \leq H$  we have a natural inclusion of  $G_H$  into  $G_F$  induced by the inclusion of  $N(H)$  into  $N(F)$ . Therefore we may regard  $G_H$  as a subgroup of  $G_F$ . Define the set

$$(5.2) \quad G_F^\circ = G_F - \bigcup_{F < H} G_H.$$

Note that  $G_F^\circ = \{\sum_{j \in \mathcal{I}(F)} a_j \lambda_j | 0 < a_j < 1\} \cap N$ , and  $G_P^\circ = G_P = \{0\}$ .

**Definition 5.1.** *We define the orbifold dolbeault cohomology groups of an omnioriented quasitoric orbifold  $\mathbf{X}$  to be*

$$H_{orb}^{p,q}(\mathbf{X}) = \bigoplus_{F \leq P} \bigoplus_{g \in G_F^\circ} H^{p-\iota(g), q-\iota(g)}(X(F)).$$

Here  $H^{p-\iota(g), q-\iota(g)}(X(F))$  refers to the components of the Hodge structures defined above, when  $X(F)$  is considered as a quasitoric orbifold  $\mathbf{X}(F)$ . The pairs  $(X(F), g)$  where  $F < P$  and  $g \in G_F^\circ$  are called twisted sectors of  $\mathbf{X}$ . The pair  $(X(P), 1)$ , i.e. the underlying space  $X$ , is called the untwisted sector.

**Definition 5.2.** We define orbifold Hodge numbers as  $h_{orb}^{p,q}(\mathbf{X}) = \dim(H_{orb}^{p,q}(\mathbf{X}))$ .

Now we introduce some notation. Consider a co-dimension  $k$  face  $F = F_1 \cap \dots \cap F_k$  of  $P$  where  $k \geq 1$ . Define a  $k$ -dimensional cone  $C_F$  in  $N \otimes \mathbb{R}$  as follows,

$$(5.3) \quad C_F = \left\{ \sum_{j=1}^k a_j \lambda_j : a_j \geq 0 \right\}.$$

The group  $G_F$  can be identified with the subset  $Box_F$  of  $C_F$ , where

$$(5.4) \quad Box_F := \left\{ \sum_{j=1}^k a_j \lambda_j : 0 \leq a_j < 1 \right\} \cap N.$$

Consequently the set  $G_F^\circ$  is identified with the subset

$$(5.5) \quad Box_F^\circ := \left\{ \sum_{j=1}^k a_j \lambda_j : 0 < a_j < 1 \right\} \cap N.$$

of the interior of  $C_F$ . We define  $Box_P = Box_P^\circ = \{0\}$ .

Suppose  $w = F_1 \cap \dots \cap F_n$  is a vertex of  $P$ . Then  $Box_w = \bigsqcup_{w \leq F} Box_F^\circ$ . This implies

$$(5.6) \quad G_w = \bigsqcup_{w \in F} G_F^\circ.$$

An almost complex orbifold is *SL* if the linearization of each  $g$  is in  $SL(n, \mathbb{C})$ . This is equivalent to  $\iota(g)$  being integral for every twisted sector. Therefore, to suit our purposes, we make the following definition.

**Definition 5.3.** An omnioriented quasitoric orbifold is said to be *quasi-SL* if the age of every twisted sector is an integer.

**5.2. Blowdowns.** In order to get a blow up of a face we replace a face by a facet with a new characteristic vector. Suppose  $F$  is a face of  $P$ . We choose a hyperplane  $H = \{\hat{p}_0 = 0\}$  such that  $\hat{p}_0$  is negative on  $F$  and  $\hat{P} := \{\hat{p}_0 > 0\} \cap P$  is a simple polytope having one more facet than  $P$ . Suppose  $F_1, \dots, F_m$  are the facets of  $P$ . Denote the facets  $F_i \cap \hat{P}$  by  $F_i$  without confusion. Denote the extra facet  $H \cap P$  by  $F_0$ .

Without loss of generality let  $F = \bigcap_{j=1}^k F_j$ . Suppose there exists a primitive vector  $\lambda_0 \in N$  such that

$$(5.7) \quad \lambda_0 = \sum_{j=1}^k b_j \lambda_j, \quad b_j > 0 \forall j.$$

Then the assignment  $F_0 \mapsto \lambda_0$  extends the characteristic function of  $P$  to a characteristic function  $\widehat{\Lambda}$  on  $\widehat{P}$ . Denote the omnioriented quasitoric orbifold derived from the model  $(\widehat{P}, \widehat{\Lambda})$  by  $\mathbf{Y}$ .

**Definition 5.4.** We define blowdown a map  $\mathbf{Y} \mapsto \mathbf{X}$  which is inverse of a blow-up. Such maps have been constructed in [8].

**Lemma 5.1.** (Lemma 4.2 [8]) If  $\mathbf{X}$  is positively omnioriented, then so is a blowup  $\mathbf{Y}$ .

**Definition 5.5.** A blowdown or blow up is called crepant if  $\sum b_j = 1$ .

**Lemma 5.2.** (Lemma 8.2 [8]) The crepant blowup of a quasi- $SL$  quasitoric orbifold is quasi- $SL$ .

**5.3. Correspondence of orbifold Hodge numbers.** The statement of the theorem we are going to prove is as follows

**Theorem 5.3.** For crepant blowdowns (or blowups) orbifold Hodge numbers of quasi- $SL$  quasitoric orbifolds do not change.

**Corollary 5.4.** For crepant resolution orbifold Hodge numbers of quasi- $SL$  quasitoric orbifolds do not change.

We admit the proof is similar to the proof of McKay Correspondence of Betti- numbers of Chen-Ruan cohomology in the author's previous paper [17] and motivated by Strong McKay Correspondence proof [2], but still we give a detailed argument for the convenience of the reader.

**5.4. Singularity and lattice polyhedron.** Following the discussion in Section 5.1, a singularity of a face  $F$  is defined by a cone  $C_F$  formed by positive linear combinations of vectors in its characteristic set  $\lambda_1, \dots, \lambda_d$  where  $d$  is the co-dimension of the face in the polytope. The elements of the local group  $G_F$  are of the form  $g = \text{diag}(e^{2\pi\sqrt{-1}\alpha_1}, \dots, e^{2\pi\sqrt{-1}\alpha_d})$ , where  $\sum_{i=1}^d \alpha_i \lambda_i \in N$ , and  $0 \leq \alpha_i < 1$ . Recall that the age

$$(5.8) \quad \iota(g) = \alpha_1 + \dots + \alpha_d.$$

is integral in quasi- $SL$  case by definition 5.3.

The singularity along the normal bundle of the sub-orbifold corresponding to interior of  $F$  is of the form  $\mathbb{C}^d/G_F$ . These singularities are same as Gorenstein toric quotient singularities in complex algebraic geometry. This means they are toric (coming from a cone)  **$SL$  orbifold singularity** ( $SL$  means linearization of a group element is  $SL$ , which in our case implies  $\iota(g)$  is integral). Now let  $N_w$  be the lattice formed by  $\{\lambda_1, \dots, \lambda_n\}$ , the characteristic vectors at a vertex  $w$  contained in the face  $F$ . Let  $m_w$  be the element in the dual lattice of  $N_w$  such that its evaluation on each  $\lambda_i$  is one. Now from Lemma 9.2 of [5] we know that the cone  $C_w$  contains an integral basis, say  $e_1, \dots, e_n$ . Suppose  $e_i = \sum a_{ij} \lambda_j$ . By (5.4)  $e_i$  corresponds to an element of  $G_w$ , and since the singularity is quasi- $SL$ ,  $\sum a_{ij}$  is integral. Hence  $m_w$

evaluated on each  $e_j$  is integral. So  $m_w$  an element of the dual of the integral lattice  $N$ .

Consider the  $(n - 1)$ -dimensional lattice polyhedron  $\Delta_w$  defined as  $\{x \in C_w \mid \langle x, m_w \rangle = 1\}$ . Note that  $\Delta_w = \{\sum_{i=1}^n a_i \lambda_i \mid a_i \geq 0, \sum a_i = 1\}$ . For any face  $F$  containing  $w$  we define  $\Delta_F = \Delta_w \cap C_F$ . If  $\{\lambda_i, \dots, \lambda_d\}$  denote the characteristic set of  $F$ , then  $\Delta_F = \{\sum_{i=1}^d a_i \lambda_i \mid a_i \geq 0, \sum a_i = 1\}$ . Hence  $\Delta_F$  is independent of the choice of  $w$ .

**Remark 5.5.** *An element  $g \in G$  of an  $SL$  orbifold singularity can be diagonalized to the form  $g = \text{diag}(e^{2\pi\sqrt{-1}\alpha_1}, \dots, e^{2\pi\sqrt{-1}\alpha_d})$ , where  $0 \leq \alpha_i < 1$  and  $\iota(g) = \alpha_1 + \dots + \alpha_d$  is integral.*

We make some definitions following [2].

**Definition 5.6.** *Let  $G$  be a finite subgroup of  $SL(d, \mathbb{C})$ . Denote by  $\psi_i(G)$  the number of the conjugacy classes of  $G$  having  $\iota(g) = i$ . Define*

$$(5.9) \quad W(G; uv) = \psi_0(G) + \psi_1(G)uv + \dots + \psi_{d-1}(G)(uv)^{d-1}.$$

**Definition 5.7.** *We define  $\text{height}(g) = \text{rank}(g - I)$ .*

**Definition 5.8.** *Let  $G$  be a finite subgroup of  $SL(d, \mathbb{C})$ . Denote by  $\tilde{\psi}_i(G)$  the number of the conjugacy classes of  $G$  having the height  $= d$  and  $\iota(g) = i$ .*

$$(5.10) \quad \widetilde{W}(G; uv) = \tilde{\psi}_0(G) + \tilde{\psi}_1(G)uv + \dots + \tilde{\psi}_{d-1}(G)(uv)^{d-1}.$$

**Definition 5.9.** *For a lattice polyhedron  $\Delta_F$  defining a  $SL$  singularity  $\mathbb{C}^d/G_F$ , we define the following:*

$$(5.11) \quad W(\Delta_F; uv) = W(G_F; uv).$$

$$(5.12) \quad \psi_i(\Delta_F) = \psi_i(G_F).$$

$$(5.13) \quad \widetilde{W}(\Delta_F; uv) = \widetilde{W}(G_F; uv).$$

$$(5.14) \quad \tilde{\psi}_i(\Delta_F) = \tilde{\psi}_i(G_F).$$

### 5.5. E-polynomial for quasitoric orbifold.

**Definition 5.10.** *We define the  $E$ -polynomial of a quasitoric orbifold  $\mathbf{X}$  as follows*

$$(5.15) \quad E_{\text{quas}}(\mathbf{X} : u, v) = \sum_{p,q} (-1)^{p+q} h^{p,q}(\mathbf{X}) u^p v^q.$$

If  $X_i$  is the stratification of the the quasitoric orbifold by inverse image of the quotient map on interior of faces  $F_i$ . **Here  $i$  runs over all the faces.**

**Theorem 5.6.**

$$(5.16) \quad E_{\text{quas}}(\mathbf{X} : u, v) = \sum_i E(X_i : u, v).$$

*Proof.* Let  $X'$  be a projective toric orbifold whose Hodge structure has been pulled backed to  $\mathbf{X}$ . The by proposition 3.4 [2] we have

$$(5.17) \quad E_{quas}(\mathbf{X} : u, v) = E(X' : u, v) = \sum_i E(X'_i : u, v).$$

Where is  $X'_i$  is stratification by inverse images of interiors of faces of the polytope of  $X'$ . Since the two orbifolds have combinatorially equivalent polytopes number of faces of a given dimension is same. And since the stratas are algebraic tori of dimension equal to its corresponding face, we can replace  $X'_i$  by the corresponding  $X_i$  in the right hand most sum. The identification of  $X'_i$  with  $X_i$  is by the combinatorial equivalence map.  $\square$

**Definition 5.11.** *We define*

$$(5.18) \quad E_{orb}(\mathbf{X} : u, v) = \sum_{p,q} (-1)^{p+q} h_{orb}^{p,q}(\mathbf{X}) u^p v^q.$$

From the above discussions and since each  $G_F$  is Abelian, it is easy to prove

$$(5.19) \quad E_{orb}(\mathbf{X} : u, v) = \sum_i E_{quas}(\overline{X}_i : u, v) \widetilde{W}(\Delta_{F_i}, uv).$$

The following can also be seen from what has been discussed in the previous subsection

$$(5.20) \quad W(\Delta_{F_i}, uv) = \sum_{X_j \geq X_i} \widetilde{W}(\Delta_{F_j}, uv).$$

$$(5.21) \quad E_{orb}(\mathbf{X} : u, v) = \sum_i E(X_i, u, v) W(\Delta_{F_i}, uv).$$

where  $X_j \geq X_i$  means  $X_i \subset \overline{X_j}$  and  $\mathbf{X}$  is a quasi- $SL$  quasitoric orbifold.

We generalize  $E_{st}$  defined in 6.7 [2] to quasitorics as it has similar stratification in to  $X'_i$ s

$$(5.22) \quad E_{st}(\mathbf{X} : u, v) = \sum_i E(X_i, u, v) W(\Delta_{F_i}, uv).$$

Comparing our  $E_{orb}$  with their  $E_{st}$  we have.

$$(5.23) \quad E_{st}(\mathbf{X} : u, v) = E_{orb}(\mathbf{X} : u, v)$$

**5.6. Proof of the main theorem.** We state the theorem again for the reader's convenience.

**Theorem 5.7.** *For crepant blowdowns(or blowups) orbifold Hodge numbers of quasi- $SL$  quasitoric orbifold do not change.*

*Proof.* Let  $\rho : \hat{\mathbf{X}} \rightarrow \mathbf{X}$  be a crepant blowdown of omnioriented quasi- $SL$  quasitoric orbifolds. We set  $\hat{X}_i := \rho^{-1}(X_i)$ . Then  $\hat{X}_i$  has a natural stratification it is enough to prove

$$(5.24) \quad E_{st}(\hat{\mathbf{X}}) = E_{st}(\mathbf{X})$$

But since quasi- $SL$  quasitoric orbifold have Gorenstein torodial singularity defined in [2] and a blow up effects only singularity cone of the face which is blowed up and neighboring cones, where things are toric and since no global patching is required, the proof of Batyrev-Dias can be imitated here.(see theorem 6.2 [2])  $\square$

## 6. APPENDIX

Here we give an argument why every simple polytope is combinatorially equivalent to a rational polytope. Every vertex  $v$  of a  $n$ -dimensional simple polytope  $P$  is the solution of  $n$  linear equations. The solution set of each equation are hyperplanes containing the codimension one sets whose intersection is the vertex. Since the coefficients form a linearly independent set, we can perturb them to get a system of coordinates which are rational and linearly independent and also the terms which are not attached to a variable can be made rational such that the new rational solution vertex and the corresponding hyperplanes forming the new codimension one faces and the faces of  $P$  which are untouched, form a polytope combinatorially equivalent to the original polytope  $P$  (since rationals are dense). Now do the same for the adjacent vertices not changing the codimension one faces already made rational. After doing this for adjacent vertices of  $v$  do it for vertices adjacent to these vertices. If at a stage we reach all vertices adjacent to a level of vertices, have gone through this adjustment, we conclude that all vertices have been adjusted. To see this if there was any vertex that has been left out we can connect one of the adjusted rational vertex to this vertex by a path of edges and since the vertex of an edge is adjacent to the other vertex of an edge, so this vertex would have received adjustment at some stage.

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